ON TOLERANCES REPRESENTABLE AS $R \circ R^-$

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Abstract. We give examples and counterexamples concerning varieties in which every tolerance is representable as $R \circ R^-$, for some reflexive and admissible relation R.

In [L] we introduced the following definitions.

Definition 1. A tolerance Θ of some algebra $\mathbf{A}$ is *representable* if and only if there exists a compatible and reflexive relation R on $\mathbf{A}$ such that $\Theta = R \circ R^-$ (here, R^- denotes the converse of R).

A tolerance Θ of some algebra $\mathbf{A}$ is *weakly representable* if and only if there exists a set K (possibly infinite) and there are compatible and reflexive relations R_k ($k \in K$) on $\mathbf{A}$ such that $\Theta = \bigcap_{k \in K}(R_k \circ R_k^-)$.

The definitions are motivated by the following Theorem from [L].

Theorem 2. *For every variety $\mathcal{V}$ and for every pair of terms p, q (of the same arity) for the language $\{\circ, \cap\}$, if p is regular, then the following are equivalent:*

(i) $\mathcal{V}$ satisfies the congruence identity $p(\alpha_1, \dots, \alpha_n) \subseteq q(\alpha_1, \dots, \alpha_n)$.

(ii) The tolerance identity $p(\Theta_1, \dots, \Theta_n) \subseteq q(\Theta_1, \dots, \Theta_n)$ holds for every algebra $\mathbf{A}$ in $\mathcal{V}$ and for all representable tolerances $\Theta_1, \dots, \Theta_n$ of $\mathbf{A}$.

(iii) The tolerance identity $p(\Theta_1, \dots, \Theta_n) \subseteq q(\Theta_1, \dots, \Theta_n)$ holds for every algebra $\mathbf{A}$ in $\mathcal{V}$ and for all weakly representable tolerances $\Theta_1, \dots, \Theta_n$ of $\mathbf{A}$.

(iv) $\mathcal{V}$ satisfies the tolerance identity $p(\Theta_1 \circ \Theta_1, \dots, \Theta_n \circ \Theta_n) \subseteq q(\Theta_1 \circ \Theta_1, \dots, \Theta_n \circ \Theta_n)$.

We say that a term p is regular if and only if in the labeled graph associated with p no vertex is adjacent with two distinct edges labeled with the same name (see [C1, C2, CD, L] for details).

The aim of the present paper is to study the notion of a (weakly) representable tolerance in more detail.

We first show that all tolerances in algebras without operations are weakly representable.

Proposition 3. *If $\mathbf{A}$ is an algebra belonging to the variety of sets (that is, an algebra without operations) then every tolerance of $\mathbf{A}$ is weakly representable.*

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Proof. Let $\mathbf{A}$ be an algebra without operations. For every pair of distinct elements $a, b \in A$ let Θ_{ab} be the reflexive and symmetric relation such that $(x, y) \in \Theta$ if and only if $\{x, y\} \neq \{a, b\}$.

Θ_{ab} is representable: define R by $x\,R\,y$ if and only if either $x = y = a$, or $x = y = b$, or $x \notin \{a, b\}$. R is clearly reflexive, and is compatible since $\mathbf{A}$ has no operations. It is easy to see that $\Theta_{ab} = R \circ R^-$.

If Θ is any tolerance of $\mathbf{A}$ then Θ is weakly representable, since $\Theta = \bigcap_{(a,b)\notin\Theta} \Theta_{ab}$. □

In contrast to Proposition 3, in algebras without operations there can be non representable tolerances. Such tolerances remain non representable if we add a certain kind of operations.

Proposition 4. *(i) In the 5-element algebra without operations there is a non representable tolerance.*

(ii) There exists a 7-element semilattice with a non representable tolerance.

(iii) There exists a 7-element algebra with a majority operation with a non representable tolerance (a majority operation *is a ternary operation f satisfying $x = f(x, x, y) = f(x, y, x) = f(y, x, x)$).*

Proof. (i) Let a, b_1, b_2, b_3, c denote the elements of the 5-element algebra without operations, and let Θ be the smallest reflexive and symmetric relation such that $a\,\Theta\,b_i$ and $b_i\,\Theta\,c$ for $i = 1, 2, 3$.

Θ is a tolerance, since the algebra has no operations, and it is easy to see that Θ is not representable. Indeed, if R is reflexive and $\Theta = R \circ R^-$ then $R \subseteq \Theta$ and $R^- \subseteq \Theta$, hence either $a\,R\,b_1$ or $b_1\,R\,a$. Suppose that $a\,R\,b_1$ (the case $b_1\,R\,a$ is similar). If $c\,R\,b_1$ then $a\,R \circ R^-\,c$, that is, $a\,\Theta\,c$, which is false, hence necessarily $b_1\,R\,c$. Continuing in the same way we obtain both $b_2\,R\,a$ and $b_3\,R\,a$, which implies $b_2\,R \circ R^-\,b_3$, hence $b_2\,\Theta\,b_3$, contradiction.

(ii) Consider the semilattice S with 6 minimal elements a, b_1, b_2, b_3, b_4, c and with a largest element 1. Let Θ be the smallest reflexive and symmetric relation such that 1 is Θ-related to all elements of S, and such that $a\,\Theta\,b_i$ and $b_i\,\Theta\,c$ for $i = 1, 2, 3, 4$.

It is easy to check that Θ is a tolerance. Suppose by contradiction that Θ is representable as $R \circ R^-$. If x, y are minimal elements of S and both $x\,R\,1$ and $y\,R\,1$, then $x\,R \circ R^-\,y$, hence $x\,\Theta\,y$. Thus $|\{x \in S | x \text{ is minimal and } x\,R\,1\}| \leq 2$, since in S there do not exist 3 pairwise Θ-connected minimal elements.

We can now repeat the arguments in (i) restricting ourselves to minimal elements x such that not $x\,R\,1$.

(iii) Consider the lattice $\langle L, +, \cdot \rangle$ with 6 atoms a, b_1, b_2, b_3, b_4, c and with a largest element 1 and a smallest element 0. If f is the ternary operation defined by $f(x, y, z) = (x + y)(x + z)(y + z)$ then $\langle L \setminus \{0\}, f \rangle$ is an algebra, since $L \setminus \{0\}$ is closed under f. We have that f is a majority operation, and the same tolerance as in (ii) is not representable. □

Even if we have showed that a majority term does not necessarily imply representability of tolerances, we can show that lattices have representable tolerances.

Proposition 5. *Suppose that the algebra* $\mathbf{A}$ *has binary terms* $\vee$ *and* $\wedge$ *such that* $\vee$ *defines a join-semilattice operation, the identities* $a \wedge (a \vee b) = a$, $(a \vee b) \wedge b = b$ *are satisfied for all elements* $a, b \in A$, *and the semilattice order induced by* $\vee$ *is a compatible relation on* $\mathbf{A}$. *Then all tolerances of* $\mathbf{A}$ *are representable.*

In particular, all tolerances in a lattice are representable.

Proof. If Θ is a tolerance of $\mathbf{A}$, let $R = \Theta \cap \leq$. R is compatible since both Θ and $\leq$ are compatible.

If $a \mathrel{\Theta} b$ then $a = a \vee a \mathrel{\Theta} a \vee b$, and $a \leq a \vee b$, thus $a \mathrel{R} a \vee b$. Similarly, $b \mathrel{R} a \vee b$, that is, $a \vee b \mathrel{R^-} b$, thus $\Theta \subseteq R \circ R^-$.

Conversely, if $(a, b) \in R \circ R^-$, say $a \mathrel{R} c \mathrel{R^-} b$, then $a \leq c$, thus $c = a \vee c$, hence $a = a \wedge (a \vee c) = a \wedge c$; similarly, $c \wedge b = b$, hence $a = a \wedge c \mathrel{\Theta} c \wedge b = b$, since both $R \subseteq \Theta$ and $R^- \subseteq \Theta$. Thus $a \mathrel{\Theta} b$. We have proved $R \circ R^- \subseteq \Theta$. $\square$

We now proceed to show that if $\mathbf{A}$ has a tolerance Θ which is not a congruence, then we can add operations to $\mathbf{A}$ in such a way that, in the expanded algebra, Θ is not even weakly representable. As a consequence, a Mal'cev condition $\mathcal{M}$ implies that every tolerance is representable if and only if $\mathcal{M}$ implies congruence permutability (Corollary 9).

Proposition 6. *Let* $\mathbf{A}$ *be any algebra, and let* Θ *be a tolerance of* $\mathbf{A}$. *Then there is an expansion* $\mathbf{A}^+$ *of* $\mathbf{A}$ *by unary operations such that* Θ *is a tolerance of* $\mathbf{A}^+$, *and any non trivial reflexive compatible relation of* $\mathbf{A}^+$ *contains* Θ.

Proof. Let $\mathbf{A}^+$ be obtained from $\mathbf{A}$ by adding, for every $a, b \in A$ such that $a \mathrel{\Theta} b$, and for every function $f : A \to \{a, b\}$, a new unary operation which represents the function. Since $a \mathrel{\Theta} b$, Θ is a tolerance of $\mathbf{A}^+$.

If R is a non trivial reflexive compatible relation of $\mathbf{A}^+$, there exist $c \neq d \in A$ such that $c \mathrel{R} d$. However, for every $a \mathrel{\Theta} b$ there is a function such that $f(c) = a$ and $f(d) = b$, thus $a = f(c) \mathrel{R} f(d) = b$, since R is compatible. This proves that $R \subseteq \Theta$. $\square$

Corollary 7. *If* $\mathbf{A}$ *is an algebra and* Θ *is a tolerance of* $\mathbf{A}$ *which is not a congruence, then there is an expansion* $\mathbf{A}^+$ *of* $\mathbf{A}$ *by unary operations such that* Θ *is a tolerance of* $\mathbf{A}^+$ *and* Θ *is not representable in* $\mathbf{A}^+$. *Actually,* Θ *is not even weakly representable in* $\mathbf{A}^+$.

Proof. Let $\mathbf{A}^+$ be an expansion of $\mathbf{A}$ as given by Proposition 6. Θ is a tolerance of $\mathbf{A}^+$ by Proposition 6; moreover, Θ is non trivial, since the trivial tolerance is a congruence. Suppose by contradiction that $\Theta = R \circ R^-$ for some reflexive and admissible relation R on $\mathbf{A}^+$, hence R and R^- are non trivial, thus $R \supseteq \Theta$ and $R^- \supseteq \Theta$, by Proposition 6. Then $\Theta = R \circ R^- \supseteq \Theta \circ \Theta$, and this implies that Θ is a congruence of $\mathbf{A}^+$, hence a congruence

of $\mathbf{A}$, contradiction. The proof that Θ is not weakly representable in $\mathbf{A}^+$ is similar. $\square$

The following result is probably known, but we give a proof, since we know no reference for it.

Proposition 8. *(a) If* $\mathbf{A}$ *is an algebra, and every tolerance of* $\mathbf{A}$ *is a congruence, then all congruences of* $\mathbf{A}$ *permute.*

(b) A variety $\mathcal{V}$ *is congruence permutable if and only if every tolerance of every algebra in* $\mathcal{V}$ *is a congruence.*

Proof. (a) If α, β are congruences of $\mathbf{A}$, let $\overline{\alpha \cup \beta}$ denote the smallest tolerance containing α and β, which is the smallest admissible relation containing $\alpha \cup \beta$. Notice that $\overline{\alpha \cup \beta} \subseteq \beta \circ \alpha$.

By assumption, $\overline{\alpha \cup \beta}$ is a congruence. Then $\alpha \circ \beta \subseteq \overline{\alpha \cup \beta} \circ \overline{\alpha \cup \beta} = \overline{\alpha \cup \beta} \subseteq \beta \circ \alpha$.

(b) is immediate from (a) and the well known result that in permutable varieties every reflexive and admissible relation is a congruence (see [HM], [S, Proposition 143]). $\square$

Trivially, every congruence α is representable, since $\alpha = \alpha \circ \alpha$. By Proposition 8(b), congruence permutability, for varieties, implies that every tolerance is representable. The next result shows that if a Mal'cev condition $\mathcal{M}$ implies that every tolerance is representable, then $\mathcal{M}$ implies congruence permutability.

Corollary 9. *Let* $\mathcal{M}$ *be either a Mal'cev condition, or a weak Mal'cev condition, or a strong Mal'cev condition. The following are equivalent:*

(i) $\mathcal{M}$ *implies congruence permutability.*

(ii) $\mathcal{M}$ *implies that every tolerance is representable.*

(iii) $\mathcal{M}$ *implies that every tolerance is weakly representable.*

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds. If $\mathcal{V}$ satisfies $\mathcal{M}$, then, by Proposition 8(b), every tolerance in every algebra in $\mathcal{V}$ is a congruence, hence is representable. Thus, (ii) holds.

(ii) $\Rightarrow$ (iii) is trivial.

We shall prove (iii) $\Rightarrow$ (i) by contradiction.

Suppose that (i) fails. Then there exists some variety $\mathcal{V}$ which satisfies $\mathcal{M}$ but which is not congruence permutable. By Proposition 8(b), there is an algebra $\mathbf{A} \in \mathcal{V}$ with a tolerance Θ which is not a congruence. By Corollary 7, $\mathbf{A}$ can be expanded to an algebra $\mathbf{A}^+$ in which Θ is a tolerance which is not weakly representable. By well known properties of Mal'cev conditions, the variety generated by $\mathbf{A}^+$ still satisfies $\mathcal{M}$, and this contradicts (iii). $\square$

Corollary 10. *(i) The class of varieties* $\mathcal{V}$ *such that every tolerance in every algebra in* $\mathcal{V}$ *is representable cannot be characterized by a weak Mal'cev condition.*

(ii) The class of varieties $\mathcal{V}$ *such that every tolerance in every algebra in* $\mathcal{V}$ *is weakly representable cannot be characterized by a weak Mal'cev condition.*

Proof. If any of those classes could be characterized by some weak Mal'cev condition $\mathcal{M}$, then, by Corollary 9, $\mathcal{M}$ would imply permutability. This is a contradiction, since Propositions 3 and 5 provide examples of non permutable varieties in which every tolerance is (weakly) representable. □

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